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LONG SURF

by

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Abstract

A digest of a mathematical investigation on surf due to long swell is offered to present the assumptions and conclusions in a form usable by the experimental physicist. The results so far obtained are mainly qualitative, but quite detailed in some respects, and radically different from the results of earlier analyses. Some new observational material is also presented. It appears that a simple, non-linear model is capable of describing the essence of the whole phenomenon — breaker formation, breaker collapse, run-up and back-wash — for a quite representative type of surf.

1. Introduction

A new mathematical treatment of surf on beaches has been given in a recent triplet of papers (Ho and Meyer, 1962, Shen and Meyer, 1963 a,b), but the presentation required to

[1]The results here presented were obtained in the course of research sponsored by the Office of Naval Research, first under contract Nonr 562(07), and later under Contract Nonr 562(34).

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prove its logical cohesion is needs addressed to the mathematical specialist. We hope to stimulate experimental work on surf by the following explanation, shorn of the technicalities, of the assumptions and the results of our analysis. Lest the theory be rejected out of hand on account of its more unexpected predictions, we present some photographs showing those to be realistic.

The analysis involves no approximations. We therefore state first all the assumptions, and then immediately all the conclusions we have drawn so far. Not all these statements can be in purely physical language, since some concepts emerge as important for which such language does not yet exist. We conclude with some tentative remarks on experimental questions which might elucidate the limitations of scope of the mathematical model.

2. Formulation.

We consider a two-dimensional motion of water on a beach of uniform slope, as for swell coming in from the sea with crests parallel to the straight shore. To avoid the additional parameters involved in the interaction of a breaker with the back-wash of the preceding wave, we restrict our attention to very long swell. More precisely, we consider a single wave traveling shoreward into water at rest.

The water motion is assumed governed by the first-order non-linear long-wave equations (Stoker 1957)

$$\partial h/\partial t + \partial(hu)/\partial x = 0 \,, \qquad (1)$$

$$\partial u/\partial t + u\partial u/\partial x + g\partial(h-h_0)/\partial x = 0 \,, \qquad (2)$$

where x denotes the horizontal distance, measured landward, t the time, $h(x,t)$ the total local water depth (Fig. 1), $h_0(x)$ the equilibrium water depth, g the gravitational acceleration, and u a horizontal water velocity (in fact, if u be interpreted as the vertical mean of the horizontal velocity component, then (1) is an exact, kinematical statement of mass conservation on a non-porous beach). We cannot at this time offer a satisfactory a-priori explanation why (2) should have any connection with water on a beach.

A second, crucial assumption is that the front of the incoming wave is formed by a bore. This term needs a clear definition in the present context. It is known (Stoker 1957) that progressive waves governed by (1), (2) form steep fronts — much as it is observed for swell approaching a beach — and then develop a singularity of the water acceleration and surface slope indicating a break-down of whatever set of assumptions underlie (1), (2). On the other hand, observation and analysis inidcate that the region in which those assumptions fail, remains of relatively short horizontal extent. A model thus becomes attractive in which any questions regarding the detailed water motion in that narrow region are ignored, it being assumed only that mass is conserved and also that momentum is conserved in the approximate sense consonant with (2). This turns out to furnish a consistent mathematical model in which the exceptional narrow region is represented by a discontinuity, the "bore", and the analysis then proceeds regardless of whether, and how, a real breaker develops.

The assumption of a bore is one of two main features distinguishing our work from that of Carrier and Greenspan (1958), who studied certain particualr, bore-free solutions of (1), (2) and showed those to have a shore behaviour radically different from that predicted under the present assumptions. Note that the distinction is not whether the waves do, or do not, form actual breakers, but whether they do, or do not, steepen locally more than is consistent with (1), (2). We are indebted to the U.S. Weather Bureau for access to unpublished computational results indicating that even very low swell is very likely to develop the singularity connected with bore formation, if it be governed by (1), (2).

The formation and early development of a bore are relatively well understood from the gasdynamical analogy (Stoker, 1957, Meyer, 1960), and the analysis here discussed therefore starts at a time when the bore is already fairly well developed and forms the front of the wave, as indicated schematically in Fig. 1. For a bore thus traveling shoreward into water at rest, the conservation conditions show (Stoker, 1957) the water level to rise from h_o on the landward side to h_b on the seaward side of the bore, and the water velocity to rise similarly from nothing to u_b, according to

$$u_b/V = 1-h_o/h_b, \qquad 2V^2 = gh_b(1+h_b/h_o), \qquad (3)$$

where V denotes the velocity of the bore itself. These conditions also imply that a certain amount of energy dissipation takes place in the bore (Stoker 1957).

Since (1), (2) represent the water motion as a strict wave-propagation process, the development of the bore over any chosen time-interval depends only on a limited part of the water motion to seaward. In this connection, the plausible assumption is made that the bore reaches the shore at a finite time. It is then shown that the water wave to seaward of the bore possesses a 'limiting ray of propagation' defined by (i) $dx/dt=u(x,t)+[gh(x,t)]^{1/2}$ and (ii) it reaches the shore at the same time as the bore. For clarity, an x,t-diagram should be consulted (lower part of Fig. 1). The time of arrival of the bore at the shore is taken as $t=0$, and the water motion is supposed to be observed at some initial time $t=T<0$, and to be predicted for $t<T$. [Of course, with this normalisation, T is one of the unknowns of the problem]. If the 'bore path' B (Fig. 1) marks the successive positions of the bore in the diagram, and the curve L marks the limiting ray, then the part of the wave that lies at $t=T$ between the bore position X and the limit ray position X_o (Fig. 1) plays the following role.

Specification of the initial wave shape $h(x,T)$ and the initial velocity distribution $u(x,T)$ from $x=X_0$ to $x=X$ is known to suffice for the determination of the bore development from $t=T$ to $t=0$ by means of (1) to (3). The wave shape to seaward of the limit ray L is irrelevant to the bore development.

A second main distinction between our work and that of Carrier and Greenspan (1958) arises in this connection. While they consider only a few, particular types of wave shapes, the assumptions here made admit a rather general class of waves, which is more likely to include those actually corresponding to observed swell. It is, in fact, one of the striking results of the analysis how very little needs to be specified in regard to the wave to seaward of the bore.

3. Selective Memory.

The work here discussed was originally undertaken to elucidate a phenomenon of importance in gas dynamics and also noted in three sample computations of Keller, Levine and Whitham (1960) on the problem outlined in the preceding section. They found three solutions of (1) to (3) with different initial behaviour to converge closely to each other with increasing time. The three solutions thus appeared to forget their initial wave shapes. The basic reason for this phenomenon was found (Ho and Meyer, 1960) in the degeneracy which (1), (2) are easily seen to experience where the water depth $h(x,t)$ vanishes. The ultimate development of the bore, close to shore , was shown to be essentially determined, not by the details of the initial wave shape and velocity distribution, but by a single qualitative property of the part of the wave extending from X_o to X (Fig. 1) at the initial time T.

The precise statement is as follows. Equations (1), (2) possess two families of rays of propagation, viz. the 'advancing' rays, of which L is a member, and the 'receding' rays, for which $dx/dt=u(x,t)-[gh(x,t)]^{1/2}$, and of which that issuing from the bore at time T will be denoted by C (Fig. 1). Instead of setting initial data from $x=X_o$ to $x=X$, it is mathematically equivalent and more convenient to set corresponding data on the segment of C extending from the bore path B to the limit ray L (Fig. 1). Ho and Meyer (1962) specify merely that the quantity

$$u(x,t) - gh_o t/x + 2[gh(x,t)]^{1/2}$$

shall be a strictly increasing function of time on this segment of C. This 'monotoneity assumption' is effectively an inequality concerning the water acceleration, and a physical interpretation is desirable. Note, however, that this assumption, while proven sufficient, is not known to be necessary for the results that follow, and hence, a simple physical interpretation is not certain to exist. Moreover, so long as no a-priori physical justification for (1), (2) is available, a similar justification for the corresponding initial conditions may be premature.

Apart from a mathematical regularity assumption of no oceanographical significance, the above specification completes the assumptions of the analysis, and we now turn to the results.

The bore height $h_b - h_o$ falls to zero, as the bore approaches the initial shore position $x=0$, and both the bore velocity V and the water velocity u_b immediately to seaward of the bore tend to a finite limit u_o (Keller, Levine and Whitham, 1960). On the basis of the monotoneity assumption, it was shown that $u_o > 0$ and in fact, that V and u_b must increase ultimately. This agrees with observation on the beach. The last stage of the bore's approach to the shore is thus a process in which potential energy is converted into kinetic energy.

It is also shown that the water acceleration, immediately to seaward of the bore, develops a singularity, as $t \rightarrow 0$, which is characterised by a parameter a_o connected with the acceleration distribution of the wave at the initial time T. This fact turns out to determine the asymptotic development of the bore, as $t \rightarrow 0$, up to about the tenth approximation. The first seven were derived explicitly, because they show in what sense surf has a selective memory. If the water height h_b on the seaward side of the bore is made non-dimensional by division by u_o^2/g and the bore position x_b is made non-dimensional by division by u_o^2/γ, where the constant

$$\gamma = -gh_o/x$$

is the beach slope in acceleration units, then the first six approximations to the relation between the two quantities are <u>independent of the initial wave shape</u> — they look like a power series with purely numerical coefficients. Only the seventh and higher approximations depend on the parameter a_o. The other properties of the bore are derivable from the relation between h_b and x_b, and their asymptotic approximations have a similar form. As the bore approaches close to the shore, it is thus seen to remember (i) the inequality for the acceleration on C, which controls the whole character of the ultimate bore behaviour, and (ii) the basic velocity scale u_o, which is an integral property of the wave forming and propelling the bore and is, presumably, a measure of the energy of the part of the wave extending from $x=X_o$ to $x=X$ (Fig. 1) at the initial time T. All details of the initial wave shape are virtually forgotten.

This result also contains a suggestion on how experimental data might be plotted profitably. If u_o and g be used to define the scales for all velocities and water heights, and u_o and γ for all horizontal distances and times, then within the framework of the assumptions stated, there is only one bore behaviour, close to shore. The more detailed effects of initial wave shape are reflected in the values of a_o and of further parameters to be anticipated in higher approximations and should, at best, be detectable only in specially designed experiments. The prediction of the influence of a_o, according to the seventh approximation, is shown in Fig. 2. Note that all the curves in this figure, though broken off for clarity, really rise to unity at $x=0$; the conversion of potential into kinetic energy is a marked effect.

4. Run-up and Back-wash.

We are indebted to Dr. Van Dorn of the Scripps Institution of Oceanography for impressing upon us the importance of an improved understanding of wave run-up in connection with tsunamis, and have therefore extended the analysis beyond the collapse of the bore on the beach. In view of the radiative character of (1), (2), it is natural that a prediction of the water motion for $t > 0$ should require a knowledge of the initial wave shape and velocity distribution for $x < X_0$ (Fig. 1). This is fully borne out in the work of Carrier and Greenspan (1958), where extending the determination of the motion over any given time-interval requires use of the initial data over a comparable, additional x-interval. But it is not at all true when a bore is present.

The reason for this is again the degeneracy of (1), (2) where $h=0$, which leads to a complicated singularity of virtually all non-observable quantities occurring in the analysis. As a result, the shore movement during the run-up and most of the back-wash, and much of the internal structure of run-up and back-wash, turn out to depend only on the same part of the initial wave shape which determines the bore's approach to the shore.

Moreover, they depend again, not on the detailed wave shape, but only on the inequality for the acceleration and on the basic velocity scale u_o.[3]

The shore line $x_s(t)$, defined by $h(x,t)=0$, is found to set out at $t=0$ to move landward with initial velocity $u_o > 0$. This implies, somewhat unexpectedly, that the shore line has a discontinuous velocity, i.e. dx_s/dt jumps from zero to u_o, at $t=0$. Plate 1 shows five frames from a film of surf; they were selected at equal intervals of 12 frames, and the man (who did not move his feet during the time covered by these five frames) provides a reference. The first three frames denoted by $t=-24$, -12 and 0, show the shore just before the arrival of a breaker, and apart from a little residual back-wash activity, the actual shore line (in contrast to the breaker) is seen to be largely at

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[3] Shen and Meyer (1963 b) proceed without approximation from the set of assumptions stated above, except that they extend the monotoneity assumption on C over an arbitrary time-interval ε beyond L (Fig. 1). But then they permit $\varepsilon \longrightarrow 0$, and since the time of intersection of C and L is an unknown of the problem, no physical extension of the earlier assumptions is involved.

rest. Comparison with the last three frames, t=0, 12 and 24, shows that, upon the arrival of the breaker at the very shore (frame t=0), the shore line assumes a considerable velocity rather suddenly. Of course, the actual process is not a discontinuous one, but as the mathematical model ignores the finer structure of the bore, so it must describe any other process on the same scale as discontinuous.

The initial acceleration of the shore line is $g\,dh_o/dx = -\gamma < 0$, and it retains this fixed deceleration during the whole run-up and part of the back-wash. In fact, what the analysis proves, under the assumptions stated, is that the limiting fluid element at the very shore – and only that element – moves independently of the rest of the water, during this period. Accordingly, the successive positions of the shore are marked in the x,t-diagram by a parabolic path P (Fig. 3); the maximum horizontal run-up distance is $u_o^2/(2\gamma)$, the corresponding run-up height above the equilibrium water level is $u_o^2/(2g)$, and at the time $t = u_o/\gamma$, the shore line begins to recede again. Of course, in view of the neglect of friction and various other effects, these quantitative predictions only furnish upper bounds for the real run-up distance, height and time. Moreover, no method is available yet for estimating the value of u_o from the properties of swell far from the shore.

The water profile close to shore, i.e. the net water height $h(x,t)$ for any fixed $t>0$ and small x_s-x, is

$$h \sim (x_s-x)^2/(3t)^2 ,$$

to the first approximation. Again, this does not apply to the very tip of the run-up and back-wash, where friction and surface tension should be important. But it indicates a much thinner sheet of run-up than might have been expected from the mathematical model, and in particular, predicts a very marked, progressive thinning of the run-up and back-wash sheet with time. It is thus not necessary to appeal entirely to seepage for an explanation of this observed effect — much of it is explainable already from the theory of ideal fluid motion on an impermeable beach.

All these predictions, though in marked contrast to those obtained from Carrier and Greenspan's particular solutions and from the linearised theories (Stoker, 1947), correspond to features visible to the casual observer on the beach. But we were startled to find that the assumptions stated above imply the presence, in the interior of the back-wash, of a singularity of the water-acceleration of the type generally associated with bore formation. The curve D marking the successive positions of the singularity in the x,t-diagram (Fig. 3) is called 'limit line'. Its genesis and precise course are not determined by the assumptions underlying the analysis, but those assumptions do imply that it must ultimately run towards $x=-\infty$, with the parabola P as asymptote, and hence its general course must be as indicated in Fig. 3.

There are only two known interpretations for a limit line (Meyer, 1960), viz. either the problem is physically irrealistic, or a bore forms. If the latter be accepted, the precise formation and development of the bore are again not determined by the assumptions of the analysis, but on general grounds (Meyer, 1960), the bore position must be to shoreward of the limit line position, even though very little so while the bore is still young and weak. The general course of the bore path in the x,t-diagram must therefore be as indicated by the broken line in Fig. 3.

Now, there are two types of limit lines, associated with different kinds of bores, and the limit line D is found to be of the type associated with a bore of the same kind as the original bore forming the front of the incoming wave. As across the original bore, therefore, the water level rises from landward to seaward across the new bore in the back-wash. But much in contrast to the original bore, the back-wash bore must be expected to move seaward rather than landward (Fig. 3). We were unable to see such a bore on our local beaches, perhaps because the swell was too short. However, Plate 2 shows three frames taken, at equal intervals of 16 frames, from a different part of the film mentioned above, and a back-wash bore with the features predicted by the analysis is clearly observed.

5. Beaches of non-uniform slope.

The analysis of (1) to (3) has not progressed nearly as far for that case, but functions $u(x,t)$ and $h(x,t)$ have been constructed (Shen and Meyer, 1963 a) which satisfy (1) to (3) approximately and possess the same type of singularity of the acceleration on the seaward side of the main bore as the functions discussed above. It is a plausible conjecture that the new functions furnish again an appropriate asymptotic approximation, and its degree of dependence on the variations of beach slope is found about as small as the degree of dependence on initial wave shape noted above.

Since the mathematical shore singularity is the same, it would be expected that the limiting fluid element at the shore will again be found to move independently of the rest of the water, during the run-up. If so, then the same upper bound $u_o^2/(2g)$ for the maximum run-up height would be obtained, provided the beach rises monotonically to landward.

6. Remarks.

As simple a mathematical model as (1) to (3) must obviously leave out of account many effects of some importance in surf. However, the observational verification of the qualitative predictions which we have presented lends strong support to a contention that the model describes the essence of at least some types of surf.

But (1) to (3) are not certain to describe all types of surf. We are indebted to Mr. Saville of the U.S. Army Beach Erosion Board for showing us films of tank experiments involving surf with a shore behaviour bearing out the predictions of (1) to (3) very roughly, but not all all to the same degree as the surf shown on Plates 1, 2. The most immediately notable difference in the circumstances was that the tank 'beach' was much steeper, and showed rapid variations of beach slope over much shorter distances, than the Californian beach of Plates 1, 2. This suggests the presumption that our model may be valid primarily for beaches which are 'flat' in some sense. Further, and especially experimental, work would seem required to clarify this point.

In particular, it remains to be determined whether islands suffering tsunami damage have slopes 'flat' with respect to tsunami swell in such a sense. In this connection, it may be relevant that it would appear not yet to be known with certainty whether all tsunami waves form bores on such island slopes. (Such local bore formation would be different from the tendency to bore formation by the very first wave noted by Miller, Munk and Snodgrass (1962) and, perhaps, explained by Shen and Meyer (1961)). Note again that the question concerns the formation of bores, not breakers, and that, even on the basis of (1), (2), it would be expected to occur only very close to shore. If bore formation does not occur, then (1), (2) would not be expected to apply to tsunami waves on such slopes, and a comparison of our predictions with experimental results might point the way to the proper mathematical model for tsunami run-up.

7. Acknowledgements.

Plates 1, 2 are from a videotape taken under the direction of Mr. B. Pennington for the Atlantic Refining Co. and made available to us though the good offices of N.W. Ayer & Sons (New York), KTTV (Hollywood) and WPRO-TV (Providence).

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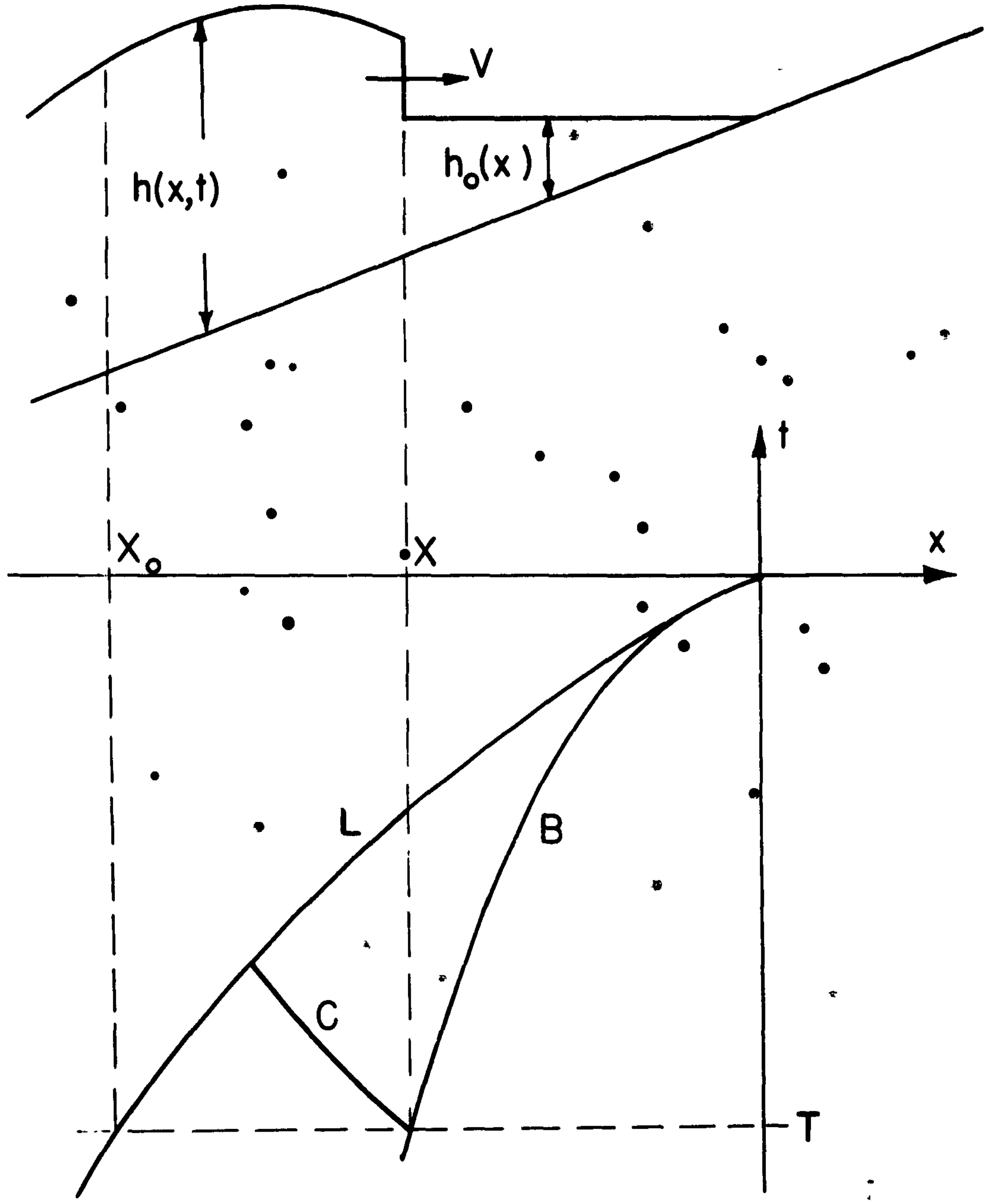
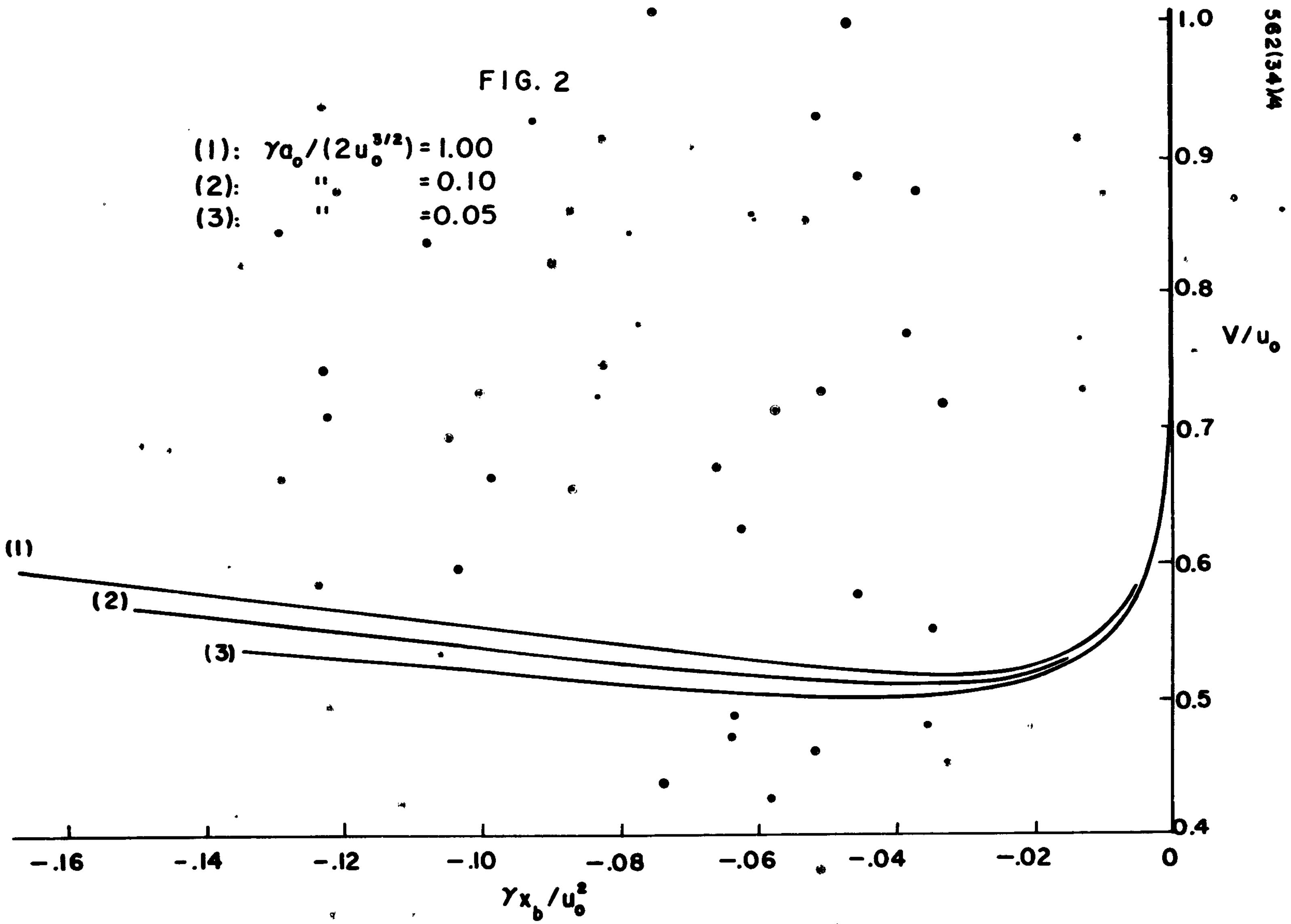


FIG. I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(1): $\gamma a_0/(2u_0^{3/2}) = 1.00$
(2): " $= 0.10$
(3): " $= 0.05$
V/u_0
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0
−.02
−.04
−.06
−.08
−.10
−.12
−.14
−.16
$\gamma x_b/u_0^2$
(1)
(2)
(3)

FIG. 2

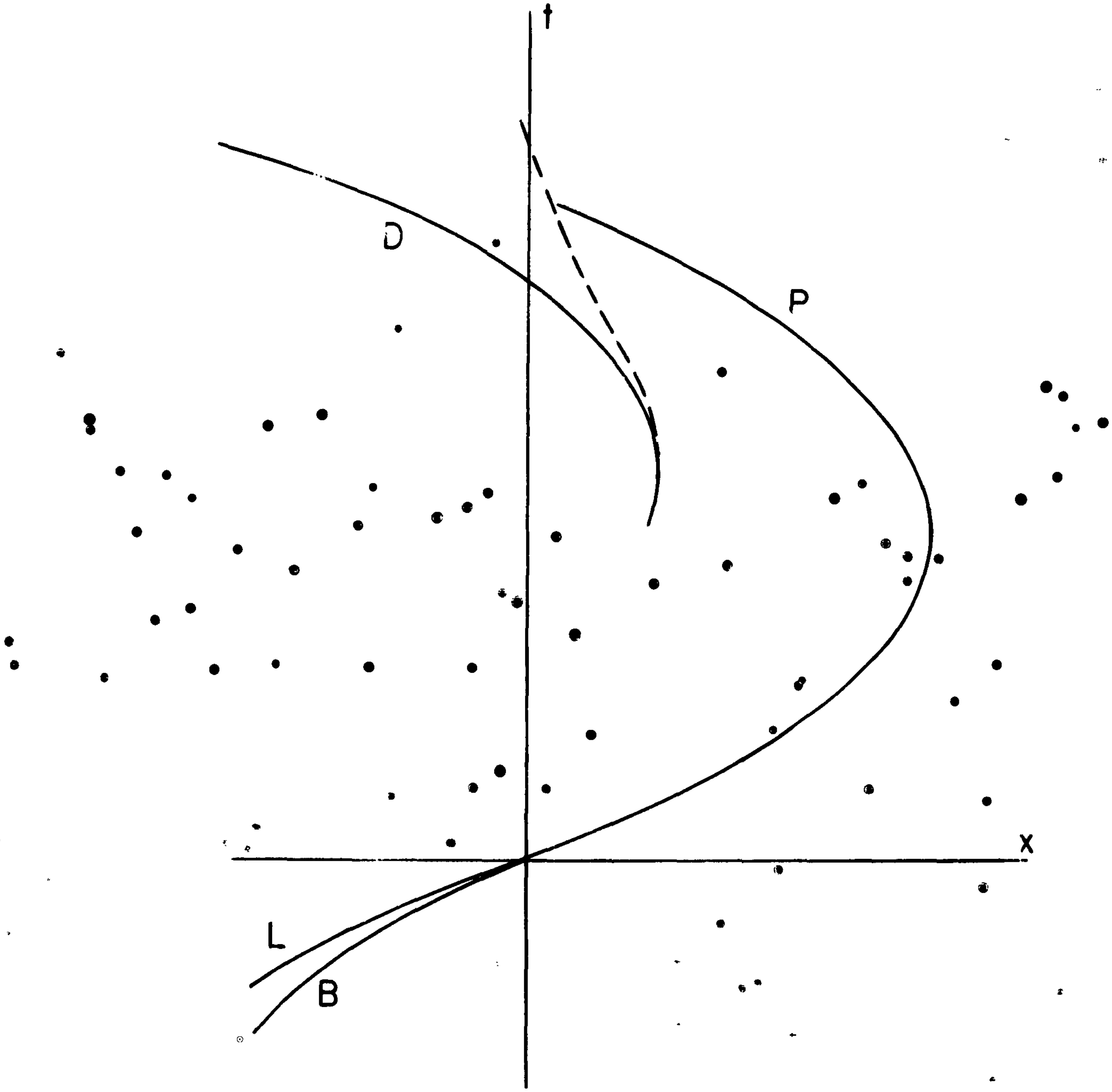
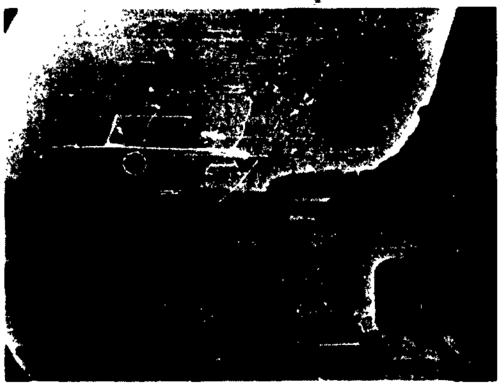


FIG. 3

t = - 24



t = - 12

t = 0

PLATE 1A

t = 0

t = 12

t = 24

PLATE IB

t = 500

t = 516

t = 532

PLATE 2